Triple Integrals

Idea: Integrate functions of 3 variables (Nothing really changes)

Now $\iiint_R f(x,y,z)\,dV$ has R a solid region in 3 space, w/ f defined thereon

Ex Compute $\iiint_E (xy+z^2)\,dV$ where $\underbrace{E=[0,2]\times[0,1]\times[0,3]}_{\text{Rectangular prism in } \mathbb{R}^3}$

Sol: $\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (xy+z^2)\,dz\,dy\,dx$

Note: We can choose order as long a we carefully reparameterize like Fubini's Theorem

$= \int_0^2\int_0^1 \left[xyz+\frac{1}{3}(z)^3\right]_0^3 dy\,dx$

$= \int_0^2\int_0^1 (3xy+9-0)\,dy\,dx$

$= \int_0^2 \left[\frac{3}{2}xy^2+9y\right]_0^1 dx \quad = \int_0^2 \frac{3}{2}x+9-0\,dx$

$= \left[\frac{3}{4}x^2+9x\right]_0^2 \quad = \frac{3}{4}(4)+18-0 = 21$ $\boxtimes$

Ex Compute $\iiint_R (2x-y)\,dV$ $R=\{(x,y,z): 0\le z\le 2,\ 0\le y\le z^2,\ 0\le x\le y-z\}$

R is parametrized w/ $\{(x,y,z): \underbrace{c_1\le z\le c_2}_{\text{constant}},\ g_1(z)\le y\le g_2(z),\ h_1(y,z)\le x\le h_2(y,z)\}$

BAD ORDER
if not reparameterized

Q: What would integral look like in dydxdz order?

$$\int_{z=0}^{2}\int_{x=0}^{y-2}\int_{y=0}^{z^2} (2x-y)\,dy\,dx\,dz$$

inner: $\int_{y=0}^{z^2} (2x-y)dy = \left[2xy - \frac{1}{2}y^2\right]_0^{z^2}$

$= 2xz^2 - \frac{1}{2}z^4 - 0$

Middle: $\int_0^{y-2} (2xz^2 - \frac{1}{2}x^4)dx = \left[x^2z^2 - \frac{1}{2}xz^4\right]_0^{y-z}$

$= \underbrace{(y-z)^2z^2 - \frac{1}{2}(y-z)z^4}_{y \text{ is constant now}}$

BAD!

Now do given order

Sol (in x, y, z order)

$$\int_{z=0}^{2}\int_{y=0}^{z^2}\int_{x=0}^{y-z} (2x-y)\,dx\,dy\,dz$$

inner $\int_{z=0}^{2}\int_{y=0}^{z^2} \left[x^2 - xy\right]_0^{y-z} dy\,dz = \int_{z=0}^{2}\int_{y=0}^{z^2} (y-z)^2 - (y-z)y\,dy\,dz$

Middle $\int_{z=0}^{2}\int_{y=0}^{z^2} z^2 - yz\,dy\,dz = \int_{z=0}^{2} \left[z^2y - \frac{1}{2}y^2z\right]_0^{z^2}$

$= \int_{z=0}^{2} z^4 - \frac{1}{2}z^5 - 0\,dz$

Outer $\int_0^2 z^4 - \frac{1}{2}z^5 - 0\,dz = \left[\frac{1}{5}z^5 - \frac{1}{12}z^6\right]_0^2 = \frac{32}{5} - \frac{64}{12}$

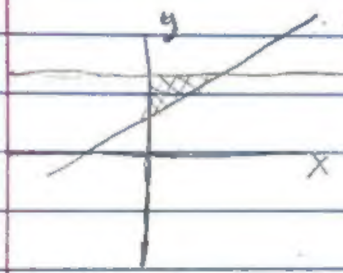
$= \boxed{16/15}$

Reparameterize bad order dydxdz

Reparameterize $\begin{cases} 0 \le z \le 2 \\ 0 \le y \le z^2 \\ 0 \le x \le y - z \end{cases}$

bounding pieces: $x=0 \quad x = y-z$
$y=0 \quad y = z^2$ → to flip, check intersections
$z=0 \quad z=2$

$\therefore$ New bounds are $R = \{(x,y,z) : 0 \le z \le 2,\ 0 \le x \le z^2 - z,\ 0 \le y \le z^2\}$



$x = y - z \,/\, y = x + z$

z^2 - constant for xy integral

$z^2 = x + z$

$\therefore x = z^2 - z$

$\therefore 0 \le x \le z^2 - z$

$\therefore \boxed{x + z \le y \le z^2}$ correct reparameterized y

Sol2: $\displaystyle\int_{z=0}^{2}\int_{x=0}^{z^2-z}\int_{y=x+z}^{z^2} (2x-y)\,dy\,dx\,dz$

inner $\displaystyle\int_{y=x+z}^{z^2} (2x-y)\,dy = \left[2xy - \tfrac{1}{2}y^2\right]_{x+z}^{z^2} = \left(2xz^2 - \tfrac{1}{2}z^4\right) - \left(2x(x+z) - \tfrac{1}{2}(x+z)^2\right)$

$= 2xz^2 - \tfrac{1}{2}z^4 - 2x^2 - 2xz + \tfrac{1}{2}x^2 + xz - \tfrac{1}{2}z^2$

$= 2xz^2 - \tfrac{1}{2}z^4 - \tfrac{3}{2}x^2 - xz + \tfrac{1}{2}z^2$

Middle $\displaystyle\int_{x=0}^{z^2-z} 2xz^2 - \tfrac{1}{2}z^4 - \tfrac{3}{2}x^2 - xz + \tfrac{1}{2}z^2\,dx$

$= \left[x^2z^2 - \tfrac{1}{2}xz^4 - \tfrac{1}{2}x^3 - \tfrac{1}{2}x^2z + \tfrac{1}{2}xz^2\right]_0^{z^2-z}$

$= (z^2-z)z^2 - \tfrac{1}{2}(z^2-z)z^4 - \tfrac{1}{2}(z^2-z)^3 - \tfrac{1}{2}(z^2-z)^2z + \tfrac{1}{2}(z^2-z)z^2 - 0$

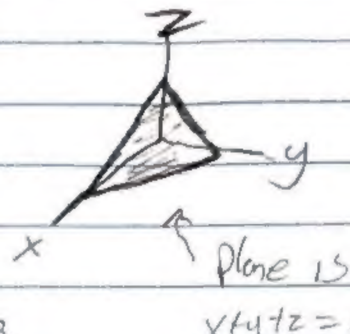
$= (z^2-z)\left((z^2-z)z - \tfrac{1}{2}z^4 - \tfrac{1}{2}(z^2-z)^2 - \tfrac{1}{2}(z^2-z)z + \tfrac{1}{2}z^2\right)$

$= (z^2-z)\left(z^4 - z^3 - \tfrac{1}{2}z^4 - \tfrac{1}{2}(z^4 - 2z^3 + z^4) - \tfrac{1}{2}z^3 + \tfrac{1}{2}z^2\right)$

$= (z^2-z)\left(-\tfrac{1}{2}z^3 - \tfrac{1}{2}z\right)$

$= -\tfrac{1}{2}z^2(z^3 + z - z^2 - 1)$ Mistake Somewhere here. Email to come later

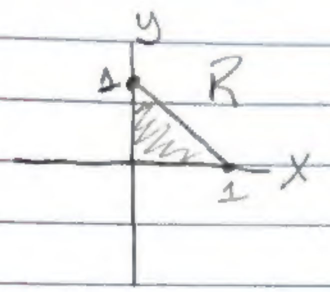
Ex Compute the volume of the tetrahedron $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$



Sol: $\text{Vol}(T) = \iiint_T 1 \, dV$

Now parameterize T

First look at possible (x,y) pairs



in xy-plane: $\{(x,y): 0 \le x \le 1, \ 0 \le y \le 1-x\}$

In T: $0 \le z \le 1-x-y$

$\therefore T = \{(x,y,z): 0 \le x \le 1, \ 0 \le y \le 1-x, \ 0 \le z \le 1-x-y\}$

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 1 \, dz\,dy\,dx$$

inner $$\int_{x=0}^{1}\int_{y=0}^{1-x} \Big[z\Big]_0^{1-x-y} dy\,dx$$

middle $$\int_{x=0}^{1}\int_{y=0}^{1-x} 1-x-y \, dy\,dx = \int_{x=0}^{1} \Big[y - xy - \tfrac{1}{2}y^2\Big]_0^{1-x} dx$$

outer $$\int_{x=0}^{1} (1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} (1-x)^2 \, dx = -\tfrac{1}{2}\cdot\tfrac{1}{3}\Big[(1-x)^3\Big]_0^1$$

$$= -\frac{1}{6}(0-1)$$

$$= \boxed{\frac{1}{6}}$$